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Study on effect of the deformation technique on the RC-distribution and its properties

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ARTICLE INFO. **ABSTRACT**

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In this paper we study the possibility of applying the deformation technique for the raised cosine distribution. This study is concerned with explaining the transformation of this distribution to other corresponding deformed distribution. Some properties and measures of the deformed distribution are deduced.

Keywords: Cosine distribution, Deformation technique, Raised cosine distribution, Reduced raised cosine distribution, Standard raised cosine distribution, Trigonometric distribution

1. Introduction

Recently, the effect of deformation technique on hyperbolic function has been studied for the hyperbolic distributions in several works (El-Shehawy (2012a, 2013b, 2017); Kyurkchiev at al (2015)). Some classes of deformed hyperbolic distributions have been proposed. Each deformed distribution has been constructed via scalar deformation parameters (parametric functions).

In regards of the trigonometric distributions, which include one or more trigonometric functions, these distributions have been discussed in several fields (Chesneau et al (2018); Karim and Seddiki (2012); Rinne (2010); Surhone and et al (2010); Ahsanullah and Shakil (2018); El-Sabbagh and et al (2009); Sinha (2012), Kyurkchiev and et al (2015)). Some classes of these distributions have been provided in various formulas. As applications, the probability density function "pdf" and the corresponding cumulative distribution function "cdf" for some of these distributions are functions commonly used to avoid inter symbol interference in communications systems (Karim and Seddiki (2009); Surhone and et al (2010); Kyurkchiev and et al (2015)).

In this paper, the goal is concerned to study effect of applying the deformation technique on raised cosine distribution "RC-distribution" as well the standard raised cosine distribution "SRC-distribution" and the reduced raised cosine distribution "RRC-distribution" as trigonometric distributions by introducing two parameters "*p* and *q*" in the interval]0, 1] on the included cosine function in these distributions. Moreover, we study the possibility of applying the deformation technique to transform RC-distribution as well its standard form and reduced form to corresponding *pq*-deformed raised cosine distribution " *pq*-DRC-distribution", the corresponding *pq*-deformed standard raised distribution " *pq*-DSRC-distribution" and *pq*-deformed reduced raised cosine distribution " *pq*-DRRC-distribution" respectively.

The rest of this paper is structured as follows: A class of symmetric cosine distributions is presented in the next section. The construction of the *pq*-deformed 3trigonometric functions with some properties is explained in section 3. Some main results about applying the *pq*-deformation technique by introducing two unequal deformation parameters in the interval]0, 1] on RC-distribution and its standard and reduced cases are given in section 4. Section 5 contains some results when two equal deformation parameters are used. Conclusions are provided in section 6.

2. Overview on a class of RC-distributions

Here, a brief survey on some trigonometric distributions involving the cosine functions is provided.

Firstly, a sample of symmetric cosine distributions in various forms is given.

The RC- distribution is one of these distributions with two parameters μ, s and with pdf in the following form:

$$f_{\mathrm{RC}}(x;\mu,s)=\frac{1}{2s}(1+\cos(\pi(\frac{x-\mu}{s})))\,;\; x\in[\mu-s,\ \mu+s]$$

, where $\mu\in R, s>0$. (1)

Moreover, the SRC-distribution has the following pdf in the standard form of RC-distribution with $\mu=0$ and $s=1$:

$$f_{\mathrm{SRC}}(x)=\frac{1}{2}(1+\cos\pi x)\;;\;\; x\in[-1,1]. \quad (2)$$

The RRC-distribution is other form of RC-distribution with $\mu=0$ and $s=\pi$ has the following pdf:

$$f_{\mathrm{RRC}}(x)=\frac{1}{2\pi}(1+\cos x)\,;\quad x\in[-\pi,\pi], \quad (3)$$

which is a reduced form of the RC-distribution.

Graphically, the following figures describe the curves of the previous pdf's (1), (2) and (3) respectively.

For more details on the these distributions with their properties see these works (Chesneau et al (2018); Karim and Seddiki (2012); Rinne (2010); Surhone and et al (2010); Ahsanullah and Shakil (2018); El-Sabbagh and et al (2009); Sinha (2012), Kyurkchiev and et al (2015)).

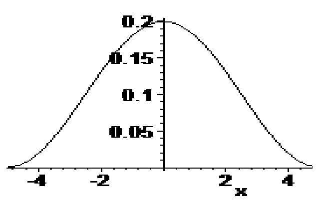

Figure 1. pdf of RC-distribution on $[-5,5]$

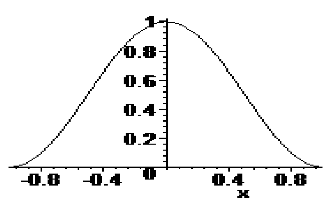

Figure 2 pdf of SRC-distribution on $[-1,1]$

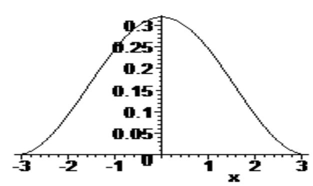

Figure 3 pdf of RRC-distribution on $[-\pi,\pi]$

3. Outline of the *pq*-deformed form for sine and cosine functions

In order to study the effect of the deformation technique on the trigonometric distributions, some main properties of deformed trigonometric functions with two scalar deformation parameters p and q, where $p,q\in]0,1]$, are given. The *pq*-deformed forms of cosine and sine functions are expressed respectively as follows:

$$\cos_{pq} x=(pe^{ix}+qe^{-ix})/2 \text{ and}$$
$$\sin_{pq} x=(pe^{ix}-qe^{-ix})/2,\ x\in R. \quad (4)$$

Each deformed function has been constructed by introducing the deformation parameters as factors of the exponential growth (positive) and decay (negative) parts of the considered trigonometric function (El-Sabbagh and et al (2009); Hassan and Abdel-Salam(2009); El-Shehawy (2012); El-Shehawy (2017)).

The *pq*-deformed cosine and sine functions satisfy the following properties:

$$\left.\begin{aligned} sin_{pq}(\pm\pi)&=i(p-q)/2,\\ cos_{pq}0-cos_{pq}\pi&=p+q,\\ cos_{pq}(\pm\pi)&=-(p+q)/2,\\ sin_{pq}(\pi/2)-sin_{pq}(-\pi/2)&=p+q,\end{aligned}\right\} \quad (5)$$

Moreover, the 1st derivatives of these *pq*-deformed trigonometric functions with respect to x can be obtained by the following relations:

$$(cos_{pq}x)'=-sin_{pq}x \text{ and } (sin_{pq}x)'=cos_{pq}x. \quad (6)$$

For more details about the deformed trigonometric functions, see (El-Sabbagh and et al (2009); Hassan and Abdel-Salam(2009);.

4. Applying the deformation technique by introducing two unequal real valued parameters on RC-distribution

In this section some main results about the possibility of applying the *pq*-deformation technique on RC-distribution and its standard and reduced cases by introducing two unequal parameters p and q in]0, 1] is provided.

The first step is to explain how the deformation technique can be applied on RC-distribution, SRC-distribution and RRC-distribution with their pdf's respectively.

For RC-distribution with two parameters μ, s, where $\mu \in R, s > 0$ and the given pdf $f_{\text{RC}}(x;\mu,s)$ in (1), by applying the *pq*-deformation technique on this pdf the following deformed form can be obtained:

$$f_{pq\text{-DRC}}(x;\mu,s,p,q) = \frac{1}{2s}(1 + cos_{pq}(\pi(\frac{x-\mu}{s}))) ;$$

$$x \in [\mu - s, \mu + s]. \tag{7}$$

Consider SRC-distribution with pdf $f_{\text{SRC}}(x)$ in (2) and applying the deformation technique via two unequal parameters p and q in $]0,1]$, the following deformed form of $f_{\text{SRC}}(x)$ can be derived:

$$f_{pq\text{-DSRC}}(x;p,q) = \frac{1}{2}(1 + cos_{pq}\pi x) \; ; x \in [-1,1]. \tag{8}$$

Finally, for RRC-distribution with pdf (3) the *pq*-deformed form of its pdf can be expressed by applying the deformation technique as follows:

$$f_{pq\text{-DRRC}}(x;p,q) = \frac{1}{2\pi}(1 + cos_{pq}\, x); x \in [-\pi,\pi], \tag{9}$$

where p and q are two unequal deformation parameters in]0, 1].

Based on these three obtained *pq*-deformed forms (7), (8) and (9), the following proposition, which refers to one of the main results, can be provided.

Proposition 1 *Via unequal real valued deformation parameters* p *and* q *in the interval]0, 1], RC-distribution on* the interval $[\mu - s, \mu + s]$ *can't be transformed to a corresponding pq-deformed distribution on the considered interval.*

Proof: Consider RC-distribution with the given pdf (1) and apply the *pq* deformation technique for $f_{\text{RC}}(x;\mu,s)$ on $[\mu - s, \mu + s]$, where $\mu \in R$, $s > 0$, the *pq*-deformed form $f_{pq\text{-DRC}}(x;\mu,s,p,q)$ in (7) can be obtained. Using (5), we can find

$$\int_{-\infty}^{\infty} f_{pq\text{-DRC}}(x;\mu,s,p,q)\, dx = \int_{\mu-s}^{\mu+s} \frac{1}{2s}(1 + cos_{pq}(\pi(\frac{x-\mu}{s})))\, dx = 1 + \frac{1}{2s}(sin_{pq}\pi - sin_{pq}(-\pi)) = 1$$

and by using MAPLE-Package, the curve of $f_{pq\text{-DRC}}(x;\mu,s,p,q)$ can't be plotted for $p \neq q$. Moreover, at $x = 0$, $\mu = 1$, $s = 4$ we find that

$$f_{pq\text{-DRC}}(0;1,4,p,q) = \frac{1}{8}(1 + cos_{pq}(-\pi/4)) = \frac{1}{8}((1 + \frac{1}{2\sqrt{2}}(p+q)) + i\frac{1}{2\sqrt{2}}(q-p))$$

is not real if $p \neq q$. This implies that, one of the two conditions of the pdf is not satisfied for $f_{pq\text{-DRC}}(x;\mu,s,p,q)$. Then, the *pq* deformation technique can't be used to transform RC-distribution (1) to a *pq*-deformed distribution if $p \neq q$.

Then, the proof is completed.

Proposition 2 *Via unequal real valued deformation parameters* p *and* q *in]0, 1]:*

(i) SRC-distribution on the $[-1,1]$ *can't be transformed to a corresponding pq-deformed distributions on the considered interval.*

(ii) RRC-distribution on $[-\pi,\pi]$ *can't be transformed to a corresponding pq-deformed distributions on the considered interval.*

Proof: Similar to the proof of proposition 1, proposition 2 can be proved as follows.

(i) Applying the deformation technique on SRC-distribution with pdf $f_{\text{SRC}}(x)$ in (2) on $[-1,1]$, then $f_{\text{SRC}}(x)$ can be given in the *pq*-deformed form (8). From (5), the definite integration of $f_{pq\text{-DSRC}}(x;p,q)$ on R equals 1, i.e.

$$\int_{-\infty}^{\infty} f_{pq\text{-DSRC}}(x;p,q)\, dx = \frac{1}{2}\int_{-1}^{1}(1 + cos_{pq}\pi x)\, dx = 1 + \frac{1}{2\pi}(sin_{pq}\pi - sin_{pq}(-\pi)) = 1.$$

On the other hand, the function $f_{pq\text{-DSRC}}(x;p,q)$ in (8) can't be graphically described for $p \neq q$ on $[-1,1]$ and moreover, at $x = 0.5$

$$f_{pq\text{-DSRC}}(1/2;p,q) = \frac{1}{2}(1 + cos_{pq}(\pi/2)) = \frac{1}{2} + i\frac{(p-q)}{4}$$

is not real if $p \neq q$. Then, one of the two conditions of the pdf is not satisfied for $f_{pq\text{-DSRC}}(x;p,q)$. This means that, the *pq* deformation technique can't be used to transform SRC-distribution (2) on $[-1,1]$ to a corresponding *pq*-deformed distribution if $p \neq q$. Then, the proof of (i) is completed.

(ii) For RRC-distribution with pdf (3), the *pq*-deformed form (9) of (3) can be obtained by applying the deformation technique. Using (5), it is clear that

$$\int_{-\infty}^{\infty} f_{pq\text{-DRRC}}(x;p,q)\,dx = \int_{-\pi}^{\pi} \frac{1}{2\pi}(1 + cos_{pq}\, x)\,dx = \frac{1}{2\pi}(2\pi + sin_{pq}\pi - sin_{pq}(-\pi)) = 1$$

,and using MAPLE-Package, the function $f_{pq\text{-DRRC}}(x;p,q)$ can't be graphically obtained for $p \neq q$ on $[-\pi,\pi]$. Moreover, at $x = \pi/4$ we find that

$$f_{pq\text{-DRRC}}(\pi/4;p,q) = \frac{1}{2\pi}(1 + cos_{pq}(\pi/4)) = \frac{1}{2\pi}((1 + \frac{1}{2\sqrt{2}}(p+q)) + i\frac{1}{2\sqrt{2}}(p-q))$$

is not real if $p \neq q$. This implies that one of the two conditions of the pdf is not satisfied for the *pq*-deformed function $f_{pq\text{-DRRC}}(x;p,q)$. Then, the *pq* deformation technique can't be used to transform RRC-distribution with the given pdf in (3) to a corresponding *pq*-deformed distribution if $p \neq q$. Then, the proof is completed.

5. Applying the deformation technique by introducing two equal real valued parameters on RC-distribution

In this section, the *pp*-deformation technique (i.e. *pq*-deformation at $p = q$) where $p \in]0,1]$, is applied. The attempts for this considered class of trigonometric distributions in the previous section can be repeated in this mention case with two real valued deformation parameters and the corresponding results are provided.

Applying the *pp*-deformation technique on the considered trigonometric distributions, then the deformed forms of (1), (2), (3) can be given respectively as follows:

$$f_{pp\text{-DRC}}(x;\mu,s,p) = \frac{1}{2s}(1 + p\cos(\pi(\frac{x-\mu}{s}))); x \in [\mu - s, \mu + s] \quad (10)$$

$$f_{pp\text{-DSRC}}(x;p) = \frac{1}{2}(1 + p\cos\pi x)\ ; \quad x \in [-1,1] \quad (11)$$

$$f_{pp-\text{DRRC}}(x;p) = \frac{1}{2\pi}(1 + p\cos x)\ ; x \in [-\pi,\pi]. \quad (12)$$

Based on these three obtained deformed forms (10), (11) and (12) respectively, the following proposition of some results and properties is given, where the following symbols can be used with the corresponding index for each considered distribution: the cdf "F", the expectation "μ", the variance "σ^2", the r^th^ non-central moment about 0 "μ_r'", the r^th^ central moment "μ_r", the skewness "γ", the kurtosis "β", the moment generating function "$M(t)$" and the characteristic function "$\Psi(t)$".

Proposition 3 *If the pp-deformation technique has been applied using* $p \in]0, 1]$ *on the considered class of the trigonometric distributions with pdf's (1), (2)* and *(3), then:*

(i) the constructed deformed form (10) of (1) can be considered as pdf of the deformed DRC-distribution "pp-DRC-distribution" with the following properties:

$$F_{pp\text{-DRC}}(x;p,\mu,s) = \frac{1}{2}(1 + \frac{x-\mu}{s} + \frac{p}{\pi}sin(\pi(\frac{x-\mu}{s}))), x \in [\mu - s, \mu + s],$$

$$\mu_{pp\text{-DRC}} = \mu'_{1,\,pp\text{-DRC}} = \mu\ ,$$

$$\mu'_{2,pp\text{-DRC}} = \mu^2 + s^2(\frac{1}{3} - \frac{2p}{\pi^2}),$$

$$\mu'_{3,pp\text{-DRC}} = \mu(\mu^2 + s^2) - 6p\mu\frac{s^2}{\pi^2},$$

$$\mu'_{4,pp\text{-DRC}} = \frac{1}{5}(5\mu^4 + 10\mu^2 s^2 + s^4) - 4p\frac{s^2}{\pi^2}((3\mu^2 + s^2) - 6\frac{s^2}{\pi^2})$$

,

$$\sigma^2_{pp\text{-DRC}} = \mu_{2,pp\text{-DRC}} = s^2(\frac{1}{3} - \frac{2p}{\pi^2}),$$

$$\mu_{3,\,pp\text{-DRC}} = \gamma_{pp\text{-DRC}} = 0\ ,$$

$$\mu_{4,pp\text{-DRC}} = \frac{s^4(\pi^4 - 20p\pi^2 + 120p)}{5\pi^4},$$

$$\beta_{pp\text{-DRC}} = -\frac{6(\pi^4 - 180p + 90p^2)}{5(\pi^2 - 6p)^2},$$

$M_{pp\text{-DRC}}(t) = \frac{\pi^2 + (1-p)s^2t^2}{st(\pi^2 + s^2t^2)}e^{\mu t}\sinh s\,t, t \neq 0$ and

$\Psi_{pp\text{-DRC}}(t) = \frac{\pi^2+(1-p)i^2s^2t^2}{st(\pi^2+i^2s^2t^2)} e^{i\mu t} \sin s\, t, |t| \neq 0, \frac{\pi}{s}$.

(ii) the constructed deformed form (11) of (2) can be considered as pdf of the deformed SRC-distribution "pp-DSRC-distribution" with the following properties:

$F_{pp\text{-DSRC}}(x;p) = \frac{1}{2}(1 + x + \frac{p}{\pi}\sin \pi x), x \in [-1,1], p \in (0,1)$

,

$\mu_{pp\text{-DSRC}} = \mu'_{1,pp\text{-DSRC}} = \mu'_{3,pp\text{-DSRC}} = \mu_{3,pp\text{-DSRC}} = \gamma_{pp\text{-DSRC}} = 0$

,

$\sigma^2_{pp\text{-DSRC}} = \mu_{2,pp\text{-DSRC}} = \mu'_{2,pp\text{-DSRC}} = \frac{1}{3} - \frac{2p}{\pi^2}$,
$\mu'_{4,pp\text{-DSRC}} = \mu_{4,pp\text{-DSRC}} = \frac{1}{5} - 4p\,\frac{1}{\pi^2}(1 - 6\frac{1}{\pi^2})$,

$\beta_{pp\text{-DSRC}} = -\frac{6(\pi^4-180p+90p^2)}{5(\pi^2-6p)^2}$,
$M_{pp\text{-DSRC}}(t) = \frac{\pi^2+(1-p)t^2}{t(\pi^2+t^2)} \sinh t, t \neq 0$ and

$\Psi_{pp\text{-DSRC}}(t) = \frac{\pi^2+(1-p)i^2t^2}{t(\pi^2+i^2t^2)} \sin t, |t| \neq 0, \pi$.

(iii) the constructed deformed form (12) of (3) can be considered as pdf of the deformed RRC-distribution "pp-DRRC-distribution" with the following properties:

$F_{pp\text{-DRRC}}(x;p) = \frac{1}{2}(1 + \frac{x}{\pi} + \frac{p}{\pi}\sin x), x \in [-\pi,\pi]$,
$\mu_{pp\text{-DRRC}} = \mu'_{1,pp\text{-DRRC}} = \mu'_{3,pp\text{-DRRC}} = \mu_{3,pp\text{-DRRC}} = 0$,

$\sigma^2_{pp\text{-DRRC}} = \mu_{2,pp\text{-DRRC}} = \mu'_{2,pp\text{-DRRC}} = \pi^2(\frac{1}{3} - \frac{2p}{\pi^2})$,

$$\mu_{4,\,pp\text{-DRRC}} = \mu'_{4,\,pp\text{-DRRC}} = \frac{1}{5}\pi^4 - 4p\ \ (\pi^2 - 6),$$

$\gamma_{pp\text{-DRRC}} = 0$, $\beta_{pp\text{-DRRC}} = -\frac{6(\pi^4-180p+90p^2)}{5(\pi^2-6p)^2}$,
$M_{pp\text{-DRRC}}(t) = \frac{1+(1-p)t^2}{\pi t(1+t^2)} \sinh \pi t, t \neq 0$ and
$\Psi_{pp\text{-DRRC}}(t) = \frac{1+(1-p)i^2t^2}{\pi t(1+i^2t^2)} \sin \pi t, |t| \neq 0, 1$.

Proof:

(i) Since $-1 \leq \cos(\pi(\frac{x-\mu}{s})) \leq 1$, then $0 \leq 1 + p\cos(\pi(\frac{x-\mu}{s}))$. This implies that $f_{pp\text{-DRC}}(x;\mu,s,p)$ is a real valued function and $f_{pp\text{-DRC}}(x;\mu,s,p) \geq 0$, $\forall p \in]0,1]$, $\forall x \in [\mu-s, \mu+s]$,. Moreover, $\mu \in R$, $s > 0$

$\int_{-\infty}^{\infty} f_{pp\text{-DRC}}(x;p)\,dx = \frac{1}{2s}\int_{\mu-s}^{\mu+s}(1 + p\cos(\pi(\frac{x-\mu}{s})))\,dx = \frac{1}{2s}(2s+0) = 1$

.

Then, there exists an effect of the deformation technique on RC-distribution and the corresponding obtained deformed distribution "*pp*-DRC-distribution" with $p \in]0,1]$ has pdf (10) on $[\mu - s, \mu + s]$. Graphically, the *pp*-deformed form (10) can be considered pdf for the *pp*-DRC-distribution only if $p \in]0,1]$ (see Fig. 4 and Fig. 5).

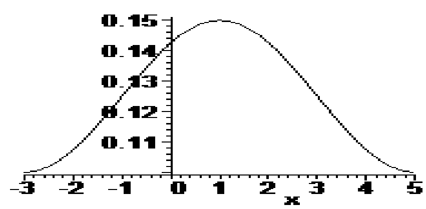

Figure 4 The *pp*-deformed form (10) of the pdf of RC-distribution on $[\mu - s, \mu + s]$ with $\mu = 1, s = 4, p = 1/5$

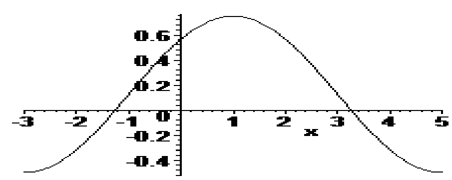

Figure 5 The pp-deformed form (10) of the pdf of RC-distribution on $[\mu - s, \mu + s]$ with $\mu = 1, s = 4, p = 5$

To derive the properties of the constructed *pp*-DRC-distribution, let X be a random variable follows the *pp*-DRC-distribution on $[\mu - s, \mu + s]$. Then

$F_{pp\text{-DRC}}(x;p) = \int_{-\infty}^{x} f_{pp\text{-DRC}}(u;p)\,du = \frac{1}{2s}\int_{\mu-s}^{x}(1 + p\cos(\pi(\frac{u-\mu}{s})))\,du$

$= \frac{1}{2}(1 + \frac{x-\mu}{s} + \frac{p}{\pi}\sin(\pi(\frac{x-\mu}{s}))), \mu - s \leq x \leq \mu + s,$

To find the non-central moments we derive the general form of the r[th] non-central moment $\mu'_{r,pp\text{-DRC}}$, where $r = 1,2,3,\ldots$, as follows:

$$\mu'_{r,pp\text{-DRC}} = \int_{-\infty}^{\infty} x^r f_{pp\text{-DRC}}(x;p)\,dx = \frac{1}{2s}\int_{\mu-s}^{\mu+s} x^r (1 + p\cos(\pi(\frac{x-\mu}{s})))\,dx$$

$$= \frac{1}{2(r+1)s}((\mu+s)^{r+1} - (\mu-s)^{r+1}) + \frac{p}{2s} I_r, r = 1,2,3,\ldots,$$

where

$$I_r = \int_{\mu-s}^{\mu+s} x^r \cos(\pi(\frac{x-\mu}{s}))\,dx = r\frac{s^2}{\pi^2}[(\mu-s)^{r-1} - (\mu+s)^{r-1}) - (r-1) I_{r-2}], r = 2,3,4,\ldots.$$

But

$$I_0 = \int_{\mu-s}^{\mu+s} \cos(\pi(\frac{x-\mu}{s}))\,dx = \frac{s}{\pi}(\sin(\pi(\frac{x-\mu}{s})))_{\mu-s}^{\mu-s} = 0,$$

$$I_1 = \int_{\mu-s}^{\mu+s} x\cos(\pi(\frac{x-\mu}{s}))\,dx = \frac{s}{\pi}(x\sin(\pi(\frac{x-\mu}{s})))_{\mu-s}^{\mu-s} + \frac{s^2}{\pi^2}(\cos(\pi(\frac{x-\mu}{s})))_{\mu-s}^{\mu-s} = 0,$$

$$I_2 = -4\frac{s^3}{\pi^2}, I_3 = -12\mu\frac{s^3}{\pi^2}, I_4 = 32\mu\frac{s^4}{\pi^2} + 48\frac{s^5}{\pi^4}.$$

This Implies that:

$$\mu_{pp\text{-DRC}} = \mu'_{1,pp\text{-DRC}} = \frac{1}{4s}((\mu+s)^2 - (\mu-s)^2) + \frac{p}{2s}I_1 = \mu,$$

$$\mu'_{2,pp\text{-DRC}} = \frac{1}{6s}((\mu+s)^3 - (\mu-s)^3) + \frac{p}{2s}I_2 = \mu^2 + s^2(\frac{1}{3} - \frac{2p}{\pi^2}),$$

$$\sigma^2_{pp\text{-DRC}} = \mu_{2,pp\text{-DRC}} = \mu'_{2,pp\text{-DRC}} - (\mu'_{1,pp\text{-DRC}})^2 = s^2(\frac{1}{3} - \frac{2p}{\pi^2}),$$

$$\mu'_{3,pp\text{-DRC}} = \frac{1}{8s}((\mu+s)^4 - (\mu-s)^4) + \frac{p}{2s}I_3 = \mu(\mu^2+s^2) - \frac{6p\mu s^2}{\pi^2},$$

$$\mu'_{4,pp\text{-DRC}} = \frac{1}{10s}((\mu+s)^5 - (\mu-s)^5) + \frac{p}{2s}I_4$$

$$= \frac{1}{5}(5\mu^4 + 10\mu^2 s^2 + s^4) - 4p\frac{s^2}{\pi^2}\left(3\mu^2 + s^2 - 6\frac{s^2}{\pi^2}\right).$$

By using the relation between the central and non-central moments, we find that

$$\mu_{3,pp\text{-DRC}} = \mu'_{3,pp\text{-DRC}} - 3\mu'_{1,pp\text{-DRC}}\mu'_{2,pp\text{-DRC}} + 2(\mu'_{1,pp\text{-DRC}})^3 = 0,$$

$$\mu_{4,pp\text{-DRC}} = \mu'_{4,pp\text{-DRC}} - 4\mu'_{1,pp\text{-DRC}}\mu'_{3,pp\text{-DRC}} + 6(\mu'_{1,pp\text{-DRC}})^2\mu'_{2,pp\text{-DRC}} - 3(\mu'_{1,pp\text{-DRC}})^4$$

$$= \frac{s^4(\pi^4 - 20p\pi^2 + 120p)}{5\pi^4}$$

$$\gamma_{pp\text{-DRC}} = \frac{\mu_{3,pp\text{-DRC}}}{(\mu_{2,pp\text{-DRC}})^{3/2}} = 0 \text{ and } \beta_{pp\text{-DRC}} = \frac{\mu_{4,pp\text{-DRC}}}{(\mu_{2,pp\text{-DRC}})^2} - 3 = -\frac{6(\pi^4 - 180p + 90p^2)}{5(\pi^2 - 6p)^2}.$$

The moment generating function and the characteristic function of X are respectively

$$M_{pp\text{-DRC}}(t) = \int_{-\infty}^{\infty} e^{tx} f_{pp\text{-DRC}}(x;p)\,dx = \frac{\pi^2 + (1-p)s^2t^2}{st(\pi^2 + s^2t^2)} e^{\mu t}\sinh st, t \neq 0$$

and

$$\Psi_{pp\text{-DRC}}(t) = \int_{-\infty}^{\infty} e^{itx} f_{pp\text{-DRC}}(x;p)\,dx = \frac{\pi^2 + (1-p)i^2s^2t^2}{st(\pi^2 + i^2s^2t^2)} e^{i\mu t}\sin st, |t| \neq 0, \frac{\pi}{s},$$

where

$$\int_{\mu-s}^{\mu+s} e^{tx}\cos(\pi(\frac{x-\mu}{s}))\,dx = \frac{-2s^2t}{(\pi^2 + s^2t^2)} e^{\mu t}\sinh st$$

and

$$\int_{\mu-s}^{\mu+s} e^{itx}\cos(\pi(\frac{x-\mu}{s}))\,dx = \frac{2s^2t}{(\pi^2 + i^2s^2t^2)} e^{i\mu t}\sin st.$$

(ii) It is clear that, the constructed deformed function $f_{pp\text{-DSRC}}(x;p)$ in (11) can be considered as pdf of *pp*-DSRC-distribution, since it is a standard case of *pp*-DRC-distribution on $[-1,1]$ where $\mu = 0$, $s = 1$, $f_{pp\text{-DSRC}}(x;p)$ is a non-negative real valued function $\forall x \in [-1,1], \forall p \in]0,1]$, and the integration of $f_{pp\text{-DSRC}}(x;p)$ on $[-\pi,\pi]$ equals 1.

This means that, there exists an effect of the deformation technique on SRC-distribution and in this case, the obtained deformed distribution "*pp*-DSRC-distribution" via two equal parameters in $]0,1]$ has pdf in the *pp*-deformed form (11).

Similar to (i), the *pp*-deformed form (11) can be graphically considered pdf for the *pp*-DSRC-distribution only if $p \in]0,1]$ (see Figure 6 and Figure 7).

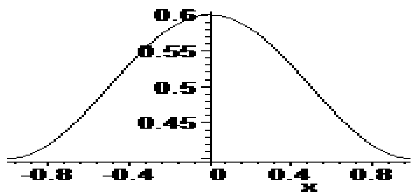

Figure 6 The *pp*-deformed form (11) of the pdf of SRC-distribution on $[-1, 1]$ with $p = 1/5$

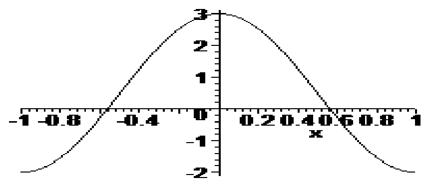

Figure 7 The *pp*-deformed form (11) of the pdf of SRC-distribution on $[-1, 1]$ with $p = 5$

The properties in (ii) of the constructed *pp*-DSRC-distribution on the interval $[-1,1]$ can be directly derived at $\mu = 0$ and $s = 1$ in the results of (i).

(iii) Similar to the case in (ii), the constructed deformed function $f_{pp\text{-DRRC}}(x\,;p)$ in (12) can be considered as pdf of *pp*-DRRC-distribution, since it is other special case of *pp*-DRC-distribution on the interval, $s = \pi$, $\mu = 0$ where $[-\pi, \pi]$ negative real valued -is a non $f_{pp\text{-DRRC}}(x\,;p)$ function $\forall x \in [-\pi, \pi]$, $\forall p \in]0,1]$, and the integration of $f_{pp\text{-DRRC}}(x\,;p)$ on $[-\pi\,,\,\pi]$ equals 1. This means that, there exists an effect of the deformation technique on RRC-distribution and in this case, the obtained corresponding deformed distribution "*pp*-DRRC-distribution" via two equal parameters in $]0,1]$ has pdf in the *pp*-deformed form (12). Moreover, the *pp*-deformed form (12) can be graphically considered pdf for the *pp*-DRRC-distribution only if $p \in]0,1]$ (see Fig. 8 and Fig. 9).

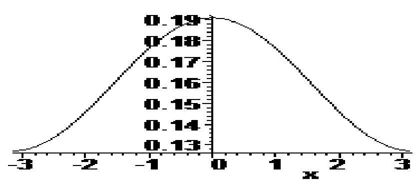

Figure 8 The *pp*-deformed form (12) of the pdf of RRC-distribution on $[-\pi, \pi]$ with $p = 1/5$

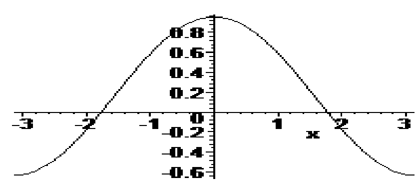

Figure 9 The pp-deformed form (12) of the pdf distribution on $[-\pi, \pi]$ with $p = 5$

Moreover, the properties of this constructed *pp*-DRRC-distribution on $[-\pi\,,\,\pi]$ can be directly obtained by using the substitution $\mu = 0$ and $s = \pi$ in the results of (i).

Then, the proof is completed.

Proposition 4 *The following properties are satisfied:*

1- For pp-DRC Distribution, where $p \in]0, 1]$:

(i) The moments $\mu'_{2,pp\text{-DRC}}$, $\mu'_{4,pp\text{-DRC}}$, $\mu_{2,pp\text{-DRC}}$ and $\mu_{4,pp\text{-DRC}}$ are monotonic decreasing functions of p ,

(ii) If $\mu > 0$ then the moment $\mu'_{3,pp\text{-DRC}}$ is a monotonic decreasing function of p and also if $\mu < 0$ then the moment $\mu'_{3,pp\text{-DRC}}$ is a monotonic increasing function of p,

(iii) The kurtosis $\beta_{pp\text{-DRC}}$ is a monotonic increasing function of p ,

(iv) The moments $\mu'_{1,pp\text{-DRC}}$, $\mu_{1,pp\text{-DRC}}$, $\mu_{3,pp\text{-DRC}}$ and the skewness $\gamma_{pp\text{-DRC}}$ are constant functions with respect to p .

2- For pp-DSRC Distribution, , where $p \in]0, 1]$:

(i) The moments $\mu'_{2,pp\text{-DSRC}}$, $\mu'_{4,pp\text{-DSRC}}$, $\mu_{2,pp\text{-DSRC}}$ and $\mu_{4,pp\text{-DSRC}}$ are monotonic decreasing functions of p ,

(ii) The moment $\mu'_{3,pp\text{-DSRC}}$ is a constant function with respect to p ,

(iii) The kurtosis$\beta_{pp\text{-DSRC}}$ is a monotonic increasing function of p,

(iv) The moments $\mu'_{1,pp\text{-DSRC}}$, $\mu_{1,pp\text{-DSRC}}$, $\mu_{3,pp\text{-DSRC}}$ and the skewness $\gamma_{pp\text{-DSRC}}$ are constant functions with respect to p .

3- For pp-DRRC Distribution, where $p \in]0, 1]$:

(i) The moments $\mu'_{2,pp\text{-DRRC}_2}, \mu'_{4,pp\text{-DRRC}}, \mu_{2,pp\text{-DRRC}}$ and $\mu_{4,pp\text{-DRRC}}$ are monotonic decreasing functions of p,

(ii) The moment $\mu'_{3,pp\text{-DRRC}}$ is a constant function with respect to p,

(iii) The kurtosisis a monotonic increasing $\beta_{pp\text{-DRRC}}$ function of p,

(iv) The moments $\mu'_{1,pp\text{-DRRC}}, \mu_{1,pp\text{-DRRC}}, \mu_{3,\ pp\text{-DRRC}}$ and the skewness $\gamma_{pp\text{-DRRC}}$ are constant function with respect to p,.

Proof: From the previous results, these properties can be directly proved as follows:

1- For *pp*-DRC Distribution, where $p \in]0,1]$:

(i) Based on the obtained forms of the 2nd central and non-central moments "$\mu_{2,\ pp\text{-DRC}}$ and $\mu'_{2,\ pp\text{-DRC}}$", we find that:

$$\mu'_{4,p_1p_1\text{-DRC}} - \mu'_{4,p_2p_2\text{-DRC}} = 4s^2((3\mu^2 + s^2) - 6s^2/\pi^2)(p_2 - p_1)/\pi^2.$$

Then

Then $\mu'_{4,p_1p_1\text{-DRC}} - \mu'_{4,p_2p_2\text{-DRC}} > 0$ if $p_2 > p_1$.

This implies that $\mu'_{2,pp\text{-DRC}}$ and $\mu_{2,\ pp\text{-DRC}}$ are monotonic decreasing functions of p.

From the obtained form of the 4th non-central moment $\mu'_{4,\ pp\text{-DRC}}$, we find that:

$$\mu_{4,p_1p_1\text{-DRC}} - \mu_{4,p_2p_2\text{-DRC}} = 4s^4(\pi^2 - 6)(p_2 - p_1)/\pi^4$$

.

Then $\mu_{4,\ p_1p_1\text{-DRC}} - \mu_{4,\ p_2p_2\text{-DRC}} > 0$ if $p_2 > p_1$.

This implies that $\mu'_{4,pp\text{-DRC}}$ is a monotonic decreasing function of p.

From the obtained form of the 4th central moment $\mu_{4,pp\text{-DRC}}$, we find that:

$$\mu_{4,p_1p_1\text{-DRC}} - \mu_{4,p_2p_2\text{-DRC}} = 4s^4(\pi^2 - 6)(p_2 - p_1)/\pi^4$$

Then $\mu_{4,p_1p_1\text{-DRC}} - \mu_{4,p_2p_2\text{-DRC}} > 0$ if $p_2 > p_1$.

This implies that $\mu_{4,\ pp\text{-DRC}}$ is a monotonic decreasing function of p.

(ii) By using the deduced forms of the 3rd non-central $\mu'_{3,\ pp\text{-DRC}}$, the following relation can be obtained:

$$\mu'_{3,p_1p_1\text{-DRC}} - \mu'_{3,p_2p_2\text{-DRC}} = 6\mu s^2(p_2 - p_1)/\pi^2.$$

Then

$$\mu'_{3,p_1p_1\text{-DRC}} - \mu'_{3,p_2p_2\text{-DRC}} = \begin{cases} +ive\text{:if} p_2 > p_1, \mu > 0 \text{or} \text{if} p_2 < p_1, \mu < 0 \\ -ive\text{:if} p_2 > p_1, \mu < 0 \text{or} \text{if} p_2 < p_1, \mu > 0 \end{cases}$$

This implies that $\mu'_{3,\ pp\text{-DRC}}$ is a monotonic decreasing function of p, where $\mu > 0$ and it is a monotonic increasing function of p, where $\mu < 0$.

(iii) From the derived form of the kurtosis $\beta_{pp\text{-DRC}}$, the following relation can be obtained:

$\beta_{p_1p_1\text{-DIC}} - \beta_{p_2p_2\text{-DIC}} = A(p_1 - p_2)$, where

$$A = \frac{6(18(\pi^4+12\pi^2)+(p_1+p_2)(1080\pi^2-54\pi^4)-648p_1p_2)}{5(\pi^2-6p_2)^2(\pi^2-6p_1)^2} > 0$$
$$,\forall p_1, p_2 \in (0,1].$$

Then $\beta_{p_1p_1\text{-DRC}} - \beta_{p_2p_2\text{-DRC}} > 0$ if $p_1 > p_2$.

This implies that $\beta_{pp\text{-DRC}}$ is a monotonic increasing function of p.

(iv) Since
$\mu'_{1,pp\text{-DRC}} = \mu, \mu_{1,pp\text{-DRC}} = \mu_{3,pp\text{-DRC}} = \gamma_{pp\text{-DRC}} = 0$,
then $\mu'_{1,\ pp\text{-DRC}}, \mu_{1,pp\text{-DRC}}, \mu_{3,\ pp\text{-DRC}}$ and $\gamma_{pp\text{-DRC}}$ are constant function with respect to p.

2- For *pp*-DSRC Distribution, where $p \in]0,1]$:

(i) Based on the obtained forms of the 2nd non-central and central moments of the *pp*-DSRC Distribution, the following relation can be obtained:

$$\mu'_{2,p_1p_1\text{-DSRC}} - \mu'_{2,p_2p_2\text{-DSRC}} = \mu_{2,p_1p_1\text{-DSRC}} - \mu_{2,p_2p_2\text{-DSRC}} = 2(p_2 - p_1)/\pi^2$$

.

Then

$\mu'_{2,p_1p_1\text{-DSRC}} - \mu'_{2,p_2p_2\text{-DSRC}} = \mu_{2,p_1p_1\text{-DSRC}} - \mu_{2,p_2p_2\text{-DSRC}} > 0$
if $p_2 > p_1$.

This implies that $\mu'_{2,\,pp\text{-DSRC}}$ and $\mu_{2,\,pp\text{-DSRC}}$ are monotonic decreasing functions of p .

Similarly, from the derived forms of the 4th non-central and central moments, the following relation can be deduced:

$\mu'_{4,p_1p_1\text{-DSRC}} - \mu'_{4,p_2p_2\text{-DSRC}} = \mu_{4,p_1p_1\text{-DSRC}} - \mu_{4,p_2p_2\text{-DSRC}} = 4(1 - 6/\pi^2)(p_2 - p_1)/\pi^2$

Then
$\mu'_{4,p_1p_1\text{-DSRC}} - \mu'_{4,p_2p_2\text{-DSRC}} = \mu_{4,p_1p_1\text{-DSRC}} - \mu_{4,p_2p_2\text{-DSRC}} > 0$
. if $p_2 > p_1$.

This implies that $\mu'_{4,pp\text{-DRRC}}$ and $\mu_{4,pp\text{-DRRC}}$ are monotonic decreasing functions of p .

(ii) Since $\mu'_{3,pp\text{-DSRC}} = 0$, then this moment is a constant function with respect to p ,

(iii) Similar to the previous proof in 1 – (iii) for *pp*-DRC distribution and from the derived form of the kurtosis $\beta_{pp\text{-DSRC}}$, the following statement can be obtained:

$\beta_{p_1p_1\text{-DSRC}} - \beta_{p_2p_2\text{-DSRC}} > 0$ if $p_1 > p_2$.

This implies that $\beta_{pp\text{-DSRC}}$ is a monotonic increasing function of p .

(iv) Since
$\mu'_{1,pp\text{-DSRC}} = \mu_{1,pp\text{-DSRC}} = \mu_{3,pp\text{-DSRC}} = \gamma_{pp\text{-DSRC}} = 0$,
then $\mu'_{1,\,pp\text{-DSRC}}$, $\mu_{1,\,pp\text{-DSRC}}$, $\mu_{3,\,pp\text{-DSRC}}$ and $\gamma_{pp\text{-DSRC}}$ are constant function with respect to p .

3- For *pp*-DRRC Distribution, where $p \in]0,1]$:

(i) Based on the obtained form of the 2nd non-central and central moments of *pp*-DRRC Distribution, the following relation can be found:

$\mu'_{2,p_1p_1\text{-DRRC}} - \mu'_{2,p_2p_2\text{-DRRC}} = \mu_{2,p_1p_1\text{-DRRC}} - \mu_{2,p_2p_2\text{-DRRC}} = 2(p_2 - p_1)$
.

Then
$\mu'_{2,p_1p_1\text{-DRRC}} - \mu'_{2,p_2p_2\text{-DRRC}} = \mu_{2,p_1p_1\text{-DRRC}} - \mu_{2,p_2p_2\text{-DRRC}} > 0$
if $p_2 > p_1$.

This means that $\mu'_{2,pp\text{-DRRC}}$ and $\mu_{2,pp\text{-DRRC}}$ are monotonic decreasing functions of. p

By using the deduced form of the 4th non-central and central moments, the following relation can be obtained:

$\mu'_{4,p_1p_1\text{-DRRC}} - \mu'_{4,p_2p_2\text{-DRRC}} = \mu_{4,p_1p_1\text{-DRRC}} - \mu_{4,p_2p_2\text{-DRRC}} = (4\pi^2 - 24)(p_2 - p_1)$.

Then

$\mu'_{4,p_1p_1\text{-DRRC}} - \mu'_{4,p_2p_2\text{-DRRC}} = \mu_{4,p_1p_1\text{-DRRC}} - \mu_{4,p_2p_2\text{-DRRC}} > 0$
if $p_2 > p_1$.

This implies that $\mu'_{4,pp\text{-DRRC}}$ and $\mu_{4,pp\text{-DRRC}}$ are monotonic decreasing functions of p .

(ii) Since $\mu'_{3,pp\text{-DRRC}} = 0$, then this moment is a constant function with respect to p,

(iii) Similar to the proof in 1 - (iii) for *pp*-DRC distribution and from the derived form of the kurtosis $\beta_{pp\text{-DRRC}}$ of *pp*-DRRC Distribution, the following statement is satisfied:

$\beta_{p_1p_1\text{-DRRC}} - \beta_{p_2p_2\text{-DRRC}} > 0$ if $p_1 > p_2$.

This implies that $\beta_{pp\text{-DRRC}}$ is a monotonic increasing function of p .

(iv) Since
$\mu'_{1,pp\text{-DRRC}} = \mu_{1,pp\text{-DRRC}} = \mu_{3,pp\text{-DRRC}} = \gamma_{pp\text{-DRRC}} = 0$,
then $\mu'_{1,\,pp\text{-DRRC}}$, $\mu_{1,\,pp\text{-DRRC}}$, $\mu_{3,\,pp\text{-DRRC}}$ and $\gamma_{pp\text{-DRRC}}$ are constant function with respect to p .

Then, the proof is completed.

6. Conclusion

Based on applying the *pq*-deformation via unequal values of the deformation parametersin]0, q and p

1], the class of RC-distribution, SRC-distribution and RRC-distribution can't be transformed to corresponding *pq*-deformed distribution on the considered intervals. On the other hand, there exists an effect of applying the *pq*-deformation technique on this class of distributions if $p = q$ and in this case, this studied class of distributions can be transformed to a corresponding class of *pp*-deformed distributions "*pp*-DRC-distribution, *pp*-DSRC-distribution and *pp*-DRRC-distribution".

Some measures and functions of the constructed *pp*-deformed distributions are derived as functions of *p*. According to the obtained results, the 2nd and 4th central or non-central moments and the kurtosis are monotonic decreasing functions of p. Moreover, the 1st central and non-central moments, the 3rd central moment and the skewness are constant functions. Although the 3rd non-central moment of *pp*-DSRC-distribution and *pp*-DRRC-distribution is a constant function, it is a monotonic decreasing (increasing) function of p for *pp*-DRC-distribution if $\mu \in R^+$ (if $\mu \in R^-$).

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